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Simplification of local energy transfer theory of incompressible, isotropic, nonstationary turbulence

R. V. R. Pandya*

Department of Mechanical Engineering, University of Puerto Rico at Mayaguez, Mayaguez, PR 00681, Puerto Rico 00680, USA

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We show that the simplification or reduction of dimension in the local energy transfer (LET) theory of incompressible, isotropic, nonstationary turbulence can be achieved by using the transitive property of the propagator with respect to intermediate times along with the supplemental equation $Q(k;t,t)=H(k;t,t')Q(k;t,t')$ and not along with the $Q(k;t,t')=H(k;t,t')Q(k;t',t')$, which has been suggested by Oberlack, McComb, and Quinn [Phys. Rev. E **63**, 026308 (2001)]. Further, we point out that the analysis presented by Oberlack *et al.* for the limiting case of viscosity approaching zero is incorrect and does not comply with the numerical solutions of LET theory.

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I. INTRODUCTION

An attempt to solve the closure problem of fluid turbulence led Kraichnan to propose the direct interaction approximation (DIA) [1,2] as a pioneer renormalized perturbation theory (RPT) followed by other RPTs which have been reviewed from time to time [3–7]. In this paper, our main concern is with local energy transfer (LET) theory of isotropic turbulence, which is compatible with the Kolmogorov spectrum [4,8]. Based on Edwards' theory [9], the LET was proposed by McComb [10] in a Eulerian framework and, since then, has evolved into a closed set of equations comprised of the fluctuation-dissipation relation and equations governing the evolution of two-time and single-time velocity correlations of isotropic turbulent flow field [8,11]. The LET has remained under persistent surveillance, especially by McComb and co-workers, for its performance and accomplishments in cases of isotropic turbulence and related passive scalar convection [11–16]. The LET's compatibility with the Kolmogorov spectrum [4] despite its failure to comply with random Galilean invariance [17], its encouraging performance, and its computational simplicity relative to some other RPTs [4,14] are certain niceties of the LET. In a recent work of Oberlack *et al.* [18], the LET equations for incompressible, isotropic, nonstationary turbulence are further simplified or reduced in dimension by incorporating solution of supplemental functional equation representing the transitive property of the propagator with respect to intermediate times, written as

$$H(k;t,t') = H(k;t,s)H(k;s,t') \quad \forall\, t \geqslant s \geqslant t' \tag{1}$$

along with one of the LET equations, namely the fluctuation-dissipation relation. Here, Eq. (1) is a scalar form of the functional equation for the propagator function $H_{ij}(\mathbf{k};t,t')$. In this paper, we show that the derivation provided by Oberlack *et al.* [18] suffers from error and their simplified equations can be obtained only after incorporating another supplemental equation,

$$Q(k;t,t) = H(k;t,t')Q(k;t,t') \quad \forall\, t \geqslant t', \tag{2}$$

where $Q(k;t,t')$ is a scalar function related to the velocity correlation $Q_{in}(\mathbf{k},-\mathbf{k};t,t')$. Further, we discuss the incorrectness in the Oberlack *et al.* work on the partial solution in the limiting case of viscosity (ν) tending to zero.

II. SIMPLIFICATION OF LET THEORY EQUATIONS

In this section, first we present the closed set of equations for LET theory for completeness, and any other equation(s) incorporated in the LET will be referred to as supplemental equation(s) or relation(s) [e.g., Eqs. (1) and (2) are supplemental equations]. Then the simplification of the LET equations will be discussed. The closed set of LET theory equations [8,11,14] consists of the generalized fluctuation-dissipation relation for the propagator $H_{in}(\mathbf{k};t,t')$, and equations governing the evolution of two-time velocity correlation $Q_{in}(\mathbf{k},\mathbf{k}';t,t')=\langle u_i(\mathbf{k},t)u_n(\mathbf{k}',t')\rangle$ and single-time velocity correlation $Q_{in}(\mathbf{k},\mathbf{k}';t,t)=\langle u_i(\mathbf{k},t)u_n(\mathbf{k}',t)\rangle$ of the velocity field $u_i(\mathbf{k},t)$ in the Fourier wave-vector–time ($\mathbf{k}$-t) domain. For isotropic turbulence, these statistical properties may be further written as

$$H_{in}(\mathbf{k};t,t') = P_{in}(\mathbf{k})H(k;t,t'), \tag{3}$$

$$Q_{in}(\mathbf{k},\mathbf{k}';t,t') = P_{in}(\mathbf{k})Q(k;t,t')\,\delta(\mathbf{k}+\mathbf{k}'), \tag{4}$$

$$Q_{\mathrm{in}}(\mathbf{k},\mathbf{k}';t,t) = P_{in}(\mathbf{k})Q(k;t,t)\,\delta(\mathbf{k}+\mathbf{k}'), \tag{5}$$

where δ represents the Dirac delta function and the projector $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_jk^{-2}$, $k=|\mathbf{k}|$, and δ_{ij} is the Kronecker delta. The LET equations for $H(k;t,t')$, $Q(k;t,t')$, and $Q(k;t,t)$ for incompressible, isotropic, nonstationary turbulence may be written as

$$Q(k;t,t') = H(k;t,t')Q(k;t',t'), \quad \forall\, t \geqslant t', \tag{6}$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)Q(k;t,t') = P(k;t,t'), \tag{7}$$

*Corresponding author. Email address: rvrpturb@uprm.edu

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q(k;t,t)=2P(k;t,t), \tag{8}$$

where the inertial transfer term $P(k;t,t')$ is

$$P(k;t,t')=\int d^3\mathbf{p}L(\mathbf{k},\mathbf{p})\left[\int_0^{t'} dsH(k;t',s)Q(p;t,s)Q(|\mathbf{k}-\mathbf{p}|;t,s)-\int_0^{t} dsH(p;t,s)Q(k;t',s)Q(|\mathbf{k}-\mathbf{p}|;t,s)\right] \tag{9}$$

and Eq. (6) represents the generalized fluctuation-dissipation relation. Also

$$L(\mathbf{k},\mathbf{p})=\frac{[\mu(k^2+p^2)-kp(1+2\mu^2)](1-\mu^2)kp}{k^2+p^2-2kp\mu} \tag{10}$$

and μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{p}$.

Here we should mention that in the derivation of the LET theory, McComb [8] invoked

$$u_i(\mathbf{k},t)=H_{in}(\mathbf{k};t,t')u_n(\mathbf{k},t'), \tag{11}$$

where

$$H_{in}(\mathbf{k};t,t')=H_{ia}(\mathbf{k};t,s)H_{an}(\mathbf{k};s,t') \quad \forall\ t \geqslant s \geqslant t' \tag{12}$$

with $H_{in}(\mathbf{k};t,t)=P_{in}(\mathbf{k})$, for the derivation of the equation for the statistically sharp propagator H_{in} whose form has been evolved into the final Eq. (6) for the homogeneous isotropic turbulence [11,14]. The supplemental Eq. (2) can be derived by multiplying Eq. (11) with $u_a(-\mathbf{k},t)$, taking the ensemble average, and using Eqs. (3)–(5). Also, the supplemental Eq. (1) can be obtained from Eq. (12) by using Eq. (3).

Simplification of the LET equations. The dimensionality of the LET equations (6)–(8) was recently reduced by Oberlack *et al.* [18] by incorporating the supplemental Eq. (1) and the LET equation (6). Oberlack *et al.* have obtained the analytical form for $H(k;t,t')$ by solving Eq. (1) as

$$H(k;t,t')=e^{h(k;t)-h(k;t')}\equiv\frac{\phi(k;t)}{\phi(k;t')}. \tag{13}$$

It should be noted that in Eq. (13), $t\geqslant t'$ and as the propagator satisfies

$$H(k;t',t)=0 \quad \forall\ t'<t, \tag{14}$$

the right-hand side (rhs) of Eq. (13) does not represent a solution for $H(k;t',t)$ when $t'<t$. Now inserting Eq. (13) into Eq. (6), we obtain

$$Q(k;t,t')=e^{h(k;t)-h(k;t')}Q(k;t',t'), \tag{15}$$

which is written by Oberlack *et al.* as

$$Q(k;t,t')=e^{h(k;t)+q(k;t')}\equiv\phi(k;t)\psi(k;t'), \tag{16}$$

where

$$e^{q(k;t')}=e^{-h(k;t')}Q(k;t',t')\equiv\frac{Q(k;t',t')}{\phi(k;t')}=\psi(k;t'). \tag{17}$$

Further, Oberlack *et al.* have invoked the symmetry in t and t' for $Q(k;t,t')$, i.e.,

$$Q(k;t,t')=Q(k;t',t), \tag{18}$$

and by substituting it in Eq. (16) they have obtained

$$\phi(k;t)\psi(k;t')=\phi(k;t')\psi(k;t). \tag{19}$$

This relation plays a crucial role in simplifying the LET equations. In fact, Oberlack *et al.* are not correct in using the symmetry property. This is because, though Eq. (18) holds true, the right-hand sides of Eqs. (15) and (16) do not possess symmetry due to the involved $H(k;t,t')$, which is not symmetric in t and t' as $H(k;t',t)=0\,\forall\,t'<t$. Also, if Oberlack *et al.* are correct, then substitution for $\phi(k;t')$ and $\psi(k;t)$ should give us $Q(k;t',t)$. But instead we obtain

$$\phi(k;t')\psi(k;t)=H(k;t',t)Q(k;t,t)=0\neq Q(k;t',t) \tag{20}$$

when $t'<t$.

Now we present correct derivation of Eq. (19). Consider the use of an additional supplemental equation or relation given by Eq. (2). This Eq. (2) along with the LET equation (6) allows us to write

$$Q(k;t,t)=H(k;t,t')H(k;t,t')Q(k;t',t') \tag{21}$$

and equivalently

$$Q(k;t,t)=\left[\frac{\phi(k;t)}{\phi(k;t')}\right]^2 Q(k;t',t'), \tag{22}$$

which yields Eq. (19) as $Q(k;t,t)/\phi(k;t)\equiv\psi(k;t)$ for any t. Thus the correct derivation of Eq. (19) requires the additional relation (2) which can be derived from Eq. (11), as mentioned earlier, but cannot be derived by using the LET equation (6) and the supplemental equation (1). Further, following Oberlack *et al.* we can write Eq. (19) as

$$\frac{\phi(k;t)}{\psi(k;t)}=\frac{\phi(k;t')}{\psi(k;t')}\equiv\frac{1}{\gamma(k)} \tag{23}$$

and the final solution for $Q(k;t,t')$ and $H(k;t,t')$ can be obtained by merging $\gamma(k)$ in $\phi(k;t)$, i.e., replacing $[\gamma(k)]^{1/2}\phi(k;t)$ by $\phi(k;t)$, written as

$$Q(k;t,t')=\phi(k;t)\phi(k;t'), \tag{24}$$

$$H(k;t,t')=\frac{\phi(k;t)}{\phi(k;t')} \quad \forall\ t'\leqslant t. \tag{25}$$

Equations (24) and (25) suggest that the solutions for $Q(k;t,t')$ and $H(k;t,t')$ can be written in terms of a single function ϕ and for which a governing equation can be obtained from the other Eqs. (7) and (8) of the LET theory. We have two Eqs. (7) and (8) to obtain an equation for one unknown ϕ. But we indicate here that Eq. (7) yields a constraint that is incorrect for the isotropic turbulence and we have to abandon the use of Eq. (7). Consequently, Eq. (8) would then generate the required equation for ϕ. Now, substituting Eqs. (24) and (25) into the LET theory Eq. (8), the equation for $\phi(k;t)$ can be written as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\phi(k;t)=\int d^3\mathbf{p}L(\mathbf{k},\mathbf{p})\int_0^t ds\frac{\phi(p;t)\phi(p;s)\phi(|\mathbf{k}-\mathbf{p}|;t)\phi(|\mathbf{k}-\mathbf{p}|;s)}{\phi(k;s)} - \int d^3\mathbf{p}L(\mathbf{k},\mathbf{p})\int_0^t ds\frac{\phi(p;t)}{\phi(p;s)}\phi(k;s)\phi(|\mathbf{k}-\mathbf{p}|;t)\phi(|\mathbf{k}-\mathbf{p}|;s). \tag{26}$$

Following Oberlack *et al.*, simplifying Eq. (7) by using Eqs. (24) and (25), and then differentiating with respect to t' yield the constraint

$$\int d^3\mathbf{p}L(\mathbf{k},\mathbf{p})\phi(p;t)\phi(p;t') \times\phi(|\mathbf{k}-\mathbf{p}|;t)\phi(|\mathbf{k}-\mathbf{p}|;t')=0 \quad \forall\, t'<t, \tag{27}$$

which suggests that the first term on the rhs of Eq. (26) vanishes. This is incorrect for isotropic turbulence, as it is not consistent with the numerical solution of the LET equations. Also the first term is responsible for quantifying the nonzero forcing term arising in the model representations for DIA [19] and Edwards's theory [9] when extended to nonsteady turbulence cases [20]. Thus we abandon Eq. (7) and constraint (27) for simplification of the LET and consider Eq. (26) as the required final equation for ϕ for the simplified local energy transfer (SLET) theory obtained from the closed set of Eqs. (1), (2), and (8).

We should mention here that, within the framework of the LET theory [8], a variant of the closed set of LET equations can be formed by replacing Eq. (6) with Eq. (2), i.e., by considering the closed set of Eqs. (2), (7), and (8). The numerical solutions of this variant of LET (VLET), SLET, and their comparative studies with the numerical solution of the LET will be considered in future work.

Now we discuss the case of $\nu\to 0$ given by Oberlack *et al.* They have provided the partial solution of the simplified LET theory equation for $\phi(k;t)$ by substituting $\nu=0$ and supposing product structure for $\phi(k;t)$ having the form

$$\phi(k;t)=f(k)g(t). \tag{28}$$

Now we show that this supposition (28) is in fact incorrect and does not provide a result consistent with the numerical solution of LET theory's equations. By substituting the supposition (28) in Eq. (25), we obtain

$$H(k;t,t')=\frac{g(t)}{g(t')} \quad \forall\, t'\leqslant t, \tag{29}$$

which is independent of k and is not correct. In fact, $H(k;t,t')$ depends on k for the LET theory as exhibited by the numerical solution of LET equations [13]. And if one assumes an exponential form (see, for example, Ref. [8])

$$H(k;t,t')=e^{-\omega_k(t-t')} \quad \forall\, t\geqslant t', \tag{30}$$

where ω_k is a function of k, then comparing it with Eq. (13) suggests a product structure for $h(k;t)$ and not for the $\phi(k;t)$.

III. CONCLUDING REMARKS

The LET theory is comprised of a set of three Eqs. (6)–(8) having three unknown dependent variables $H(k;t,t')$, $Q(k;t,t')$, and $Q(k;t,t)$. Using Eq. (6), we can eliminate $H(k;t,t')$ from Eqs. (7) and (8), thus reducing the dimensionality in the dependent variables to two variables $Q(k;t,t')$ and $Q(k;t,t)$. We have shown that the dimensionality can be further reduced to one variable $\phi(k;t)$ by incorporating supplemental Eqs. (1) and (2) and not by incorporating Eq. (1) and using symmetry property for $Q(k;t,t')$ along with the fluctuation-dissipation relation (6) as suggested by Oberlack *et al.* [18]. It should be noted that incorporation of the supplemental equations results in Eqs. (24) and (25), which suggest that $Q(k;t,t')$ and $H(k;t,t')$ are always positive quantities for any values of k, t, and t' as $\phi(k;t)=[\gamma(k)]^{1/2}e^{h(k;t)}\geqslant 0$, whereas numerical solutions for LET equations exhibit negative values for $Q(k;t,t')$ and $H(k;t,t')$ for certain values of k, t, and t' [13].

[1] R. H. Kraichnan, Phys. Rev. **109**, 1407 (1958).
[2] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).
[3] D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon Press, Oxford, 1973).
[4] W. D. McComb, *The Physics of Fluid Turbulence* (Oxford University Press, New York, 1990).
[5] W. D. McComb, Rep. Prog. Phys. **58**, 1117 (1995).
[6] V. S. L'vov, Phys. Rep. **207**, 1 (1991).
[7] M. Lesieur, *Turbulence in Fluids*, 3rd ed. (Kluwer, Dordrecht, 1997).
[8] W. D. McComb, J. Phys. A **11**, 613 (1978).
[9] S. F. Edwards, J. Fluid Mech. **18**, 239 (1964).
[10] W. D. McComb, J. Phys. A **7**, 632 (1974).
[11] W. D. McComb, M. J. Filipiak, and V. Shanmugasundaram, J. Fluid Mech. **245**, 279 (1992).
[12] W. D. McComb and V. Shanmugasundaram, J. Fluid Mech. **143**, 95 (1984).
[13] W. D. McComb, V. Shanmugasundaram, and P. Hutchinson, J. Fluid Mech. **208**, 91 (1989).
[14] W. D. McComb and A. P. Quinn, Physica A **317**, 487 (2003).
[15] J. S. Frederiksen, A. G. Davies, and R. C. Bell, Phys. Fluids **6**, 3153 (1994).
[16] J. S. Frederiksen and A. G. Davies, Geophys. Astrophys. Fluid Dyn. **92**, 197 (2000).
[17] R. H. Kraichnan, Phys. Fluids **8**, 575 (1965).
[18] M. Oberlack, W. D. McComb, and A. P. Quinn, Phys. Rev. E **63**, 026308 (2001).
[19] R. H. Kraichnan, J. Fluid Mech. **41**, 189 (1970).
[20] R. H. Kraichnan, J. Fluid Mech. **47**, 513 (1971).